Shifted Symmetric Functions and Young Tableaux

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We provide an expository overview of shifted symmetric functions by focusing on their evaluation on Young diagrams and their expression in multirectangular coordinates. Shifted Schur functions are shown to be polynomials in these coordinates, with nonnegative coefficients when expanded in the falling factorial basis. Motivated by classical results on symmetric functions, we then explore relations and generalizations to Jack polynomials. This paper surveys the work of Alexandersson and Féray (2017) and the frameworks provided by Stanley (2001, 2002) and Okounkov–Olshanski (1997a, 1997b).

Introduction

1.1 Motivation

Shifted symmetric functions arise as deformations of classical symmetric functions, where symmetry is considered with respect to the shifted variables. Their study, initiated by Okounkov and Olshanski (1997b), revealed connections with structures from classical symmetric function theory, including Schur functions, Kostka numbers, and character formulas.

While classical symmetric functions are invariant under arbitrary permutations of variables, shifted symmetric functions are formal power series symmetric in the shifted variables. Such functions can be evaluated on Young diagrams, which establish a relation between algebraic combinatorics and the representation theory of symmetric groups.

Among shifted symmetric functions, the shifted Schur functions $S^\star_\mu$ and the functions K_μ related to shifted Kostka numbers are of interest. Both are determined by their evaluations on Young diagrams in multirectangular coordinates and have nonnegative coefficients when expressed in terms of falling factorials.

We introduce multirectangular coordinates through the work of Stanley (2001, 2002). He describes a Young diagram via two sequences of non-negative integers, $(p_1, \cdots, p_d)$ and $(r_1, \cdots, r_d)$. Expressing shifted symmetric functions in such coordinates gives positivity.

The paper aims to provide an expository overview of the theory of shifted symmetric functions and to study their polynomial structure in multirectangular coordinates. We also discuss its relations to Jack polynomials (Lassalle, 2009; Goulden & Jackson, 1996).

1.2 Format

In Section 2, we define shifted symmetric functions and their connection with Young diagrams, allowing us to introduce the shifted Schur functions $S^\star_\mu$, following the constructions of (Okounkov & Olshanski, 1997a). In Section 3, we follow Stanley's work (2002) in defining set partitions and multirectangular coordinates in order to prove the shifted Schur function has nonnegative rational coefficients in the falling factorial basis (Alexandersson & Féray, 2017, Theorem 1.2). Section 4 applies a similar proof but to the shifted Jack polynomial (Alexandersson & Féray, 2017, Theorem 1.5).

1.3 Main Results

We summarize here the main results of this paper. The precise definitions of notation will be provided in later sections.

- Theorem 3.1 (Stanley, 2001; Okounkov & Olshanski, 1997b): If F is an α-shifted function, as in Definition 2.1, then when evaluated on a Young diagram λ, its value F_λ, expressed in multirectangular coordinates (p_i, r_i)(see Definition 3.5) is a polynomial in (p_i, r_i) with coefficients in $\mathbb{Q}(\alpha)$.

1	2	4	7
3	5	8	
6			

Figure 1. An example of a standard Young tableau of shape $\lambda = (4,3,1)$.

- Theorem 3.2 (Alexandersson & Féray, 2017, Theorem 1.2): For each partition μ, the shifted Schur function, $S^\star_\mu\left(\mathbf{r}^p\right)$, as defined in Definition 2.2, admits an expansion with non-negative rational coefficients in the falling factorial basis $\mathfrak{B}$ (see Definition 4.2).
- Theorem 4.1 (Alexandersson & Féray, 2017, Theorem 1.5): For every integer k, the shifted Jack polynomial, denoted $J^{\star,(\alpha)}_{(k)}\left(\mathbf{r}^p\right)$, given in Definition 4.1, has non-negative rational coefficients when expressed in the α-falling factorial basis.

Shifted Symmetric Functions and Young Tableaux

2.1 Shifted Symmetric Functions

Definition 2.1 (Shifted Symmetric Function) (Alexandersson & Féray, 2017).An α-shifted symmetric function F is a sequence $(F_N)_{N\geq 1}$ such that

1. For each $N \geq 1$, is F_n a polynomial in N variables $(x_1, \cdots, x_N)$ with coefficients in $\mathbb{Q}(\alpha)$ that is symmetric in $x_1 - \frac{1}{\alpha}, x_2 - \frac{2}{\alpha}, \cdots, x_N - \frac{N}{\alpha}$.
2. For each $N \geq 1$, $F_{N+1}(x_1, \cdots, x_N, 0) = F_N(x_1, \cdots, x_N)$(this is known as the stability property.)
3. $\sup_{N\geq 1} \deg(F_N) < \infty$.

Definition 2.2 (Shifted Schur Function). For any $\mathrm{Par} \ni \mu$ and $n \geq 1$, we define the shifted Schur function $S^\star_\mu$ in variables $(x_1, \cdots, x_n)$ by:

$$S^\star_\mu(x_1, \ldots, x_n) = \frac{\det\left((x_i + n - 1)_{\mu_j + n - j}\right)}{\det\left((x - i + n - i)_{n-j}\right)}$$

We note that the highest degree terms of $S^\star_\mu$ is exactly the Schur function S_μ. In addition to Subsection 2.3, we can observe that well-known formulas involving Schur functions naturally extend themselves to shifted Schur functions. Such formulas include Jacobi-Trudi identity and the combinatorial expansion in terms of semi-Standard Young tableaux (Okounkov & Olshanski, 1997a).

The above serves as an example of a 1-shifted symmetric function.

2.2 Young Tableaux

Define a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ to be a finite, weakly decreasing sequence of non-negative integers. The associated Young diagram is a left-justified array of boxes, with λ_i boxes in the *i*-th row from the top. We identify a partition with its Young diagram.

Definition 2.3 (Standard Young Tableau). A standard Young tableau of shape λ is a filling of the Young diagram of λ with numbers $1, 2, \ldots, |\lambda|$ such that entries increase along each row (left to right) and increase down each column (top to bottom). For example, the following is a standard Young tableau of shape $\lambda = (4, 3, 1)$.

Definition 2.4 (Semi-standard Young Tableau). A semi-standard Young tableau (SSYT) of shape λ is a filling of the Young diagram with positive integers such that entries weakly increase along each row and strictly increase down each column. We define the type of an SSYT to be the weak composition $\tau = (\tau_1, \tau_2, \ldots)$ where τ_i counts the number of boxes filled with the integer *i*.

We then have that the Kostka number K^λ_τ counts the number of SSYTs of shape λ and type τ (Macdonald, 1995).

2.3 Shifted Symmetric Functions and Tableaux

The connection between shifted symmetric functions and Young tableaux yields results that do not occur in the non-shifted setting. In fact, the evaluation $S^\star_\mu(\lambda)$ of a shifted Schur function equals 0 unless λ contains μ. When λ contains μ, we find $S^\star_\mu(\lambda)$ counts, up to normalization, the number of standard tableaux of skew shape λ/μ (Okounkov & Olshanski, 1997a).

As $S^\star_\mu(\lambda)$ and $K_\mu(\lambda)$ have tableau interpretations, we can understand shifted symmetric functions as polynomial versions of tableau counting. Moreover, if we express Young diagrams using multirectangular coordinates, we can show that shifted symmetric functions become polynomials in multirectangular coordinates with positivity properties.

Multirectangular Coordinates and Shifted Schur Functions

3.1 Set Partitions

We refer to Section 3.1 of (Alexandersson & Féray, 2017) for the definitions of set partitions.

Definition 3.1 (Set Partition). A set partition of a finite set E is a collection of non-empty, pairwise disjoint subsets $S = \{S_1, S_2, \ldots, S_s\}$ whose union is E. Each subset S_i is called a block and the number of blocks is given by $s = |S|$.

The order of set partitions is defined by the *refinement order*. Given two partitions of E denoted $\widetilde{S}$ and S, we say $\widetilde{S} \leq S$ if every block of $\widetilde{S}$ is contained entirely within some block of S. In this case, we write $\widetilde{S}$ as finer than S, and S is coarser than $\widetilde{S}$.

Remark 3.1. The set of all partitions of E, ordered by refinement, forms a lattice. In this lattice, any two partitions have a greatest lower bound (their common refinement) and a least upper bound (their common coarsening).

Definition 3.2 (Induced Partition). Let $\widetilde{S} \leq S$ be partitions of E, and let S_i be a block of S. The induced partition $S/\widetilde{S}_i$ is the set partition formed by the blocks of $\widetilde{S}$ that are contained in S_i.

Intuitively, we may see that within each block S_i of S, the partition $\widetilde{S}$ may split S_i into smaller pieces. Such pieces form a partition of S_i.

Proposition 3.1. Given a partition S of E, if for each block $S_i$we choose a partition $\widetilde{S}_i$ of S_i, then the union of the partitions $\widetilde{S}_i$ forms a partition $\widetilde{S}$ of E that is finer than S.

Conversely, given a partition $T = \{T_1, \cdots, T_t\}$ of the blocks of S, we can form a coarser partition of E by taking unions,

$$T_j \mapsto \bigcup_{S_i \in T_j} S_i.$$

Thus, partitions of the set of blocks correspond naturally to partitions of E that are coarser than S.

We now introduce how functions relate to set partitions.

Definition 3.3 (Functions and Induced Partitions). Let S be a partition of E and let $f : S \to D$ be a function assigning a value in D to each block of S. This induces a function $\bar{f} : E \to D$by setting $\bar{f}(e) = f(S_i)$ if $e \in S_i$.

Definition 3.4 (Partitions from Functions). Conversely, given any function $g : E \to D$, we define the associated set partition

$$\Pi(g) = \{g^{-1}(\{x\}) : x \in g(E)\},$$

where two elements e, e' are in the same block of $\Pi(g)$ if and only if $g(e) = g(e')$.

Proposition 3.2. Let S be a set partition of E and let $f : S \to D$ be a function. Then:

$$S \leq \Pi\left(\bar{f}\right),$$

meaning that S is finer than the partition induced by $\bar{f}$.

3.2 Multirectangular Coordinates

Multirectangular coordinates were introduced by Stanley (2001, 2002) and provide a natural way to parametrize Young diagrams. This coordinate system allows us to express evaluations as explicit polynomials in combinatorial parameters associated with the shape of the tableau. This pays dividends when working with shifted symmetrical functions.

Definition 3.5 (Multirectangular Coordinates). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. The multirectangular coordinates of λ consists of two sequences of non-negative integers

$$(p_1, \ldots, p_d), \quad (r_1, \ldots, r_d)$$

satisfying the following:

- $p_i > 0$ for all i;
- $r_1 \geq r_2 \geq \ldots \geq r_d \geq 0$;
- The rows of λ are grouped into d blocks with

$$\lambda = \left(\underbrace{r_1 + r_2 + \ldots + r_d}_{p_1 \quad \text{times}}, \underbrace{r_2 + r_3 + \ldots + r_d}_{p_2 \quad \text{times}}, \ldots, \underbrace{r_d}_{p_d \quad \text{times}} \right).$$

If we require each block to group together consecutive rows of equal length and that each grouping is maximal (i.e., no two consecutive blocks can be merged without violating the decreasing width condition, then our multirectangular coordinates are uniquely determined.

Example 3.1. Take $\lambda = (5, 3, 3, 2)$. The first row has 5 boxes, the second and third row each has 3 boxes, and the fourth row has 2 boxes. This gives the multirectangular decomposition of $(p_1, p_2, p_3) = (1, 2, 1)$ and $(r_1, r_2, r_3) = (2, 1, 2)$. Graphically, the Young diagram can be visualized as the combination of three rectangles.

Let us now consider several properties of multirectangular coordinates.

Lemma 3.1 (Uniqueness). The multirectangular decomposition of a Young tableau under the maximal grouping convention is unique.

Proof. Given a Young diagram, we proceed bottom-up. Start with the bottommost rows, identifying the maximal set of consecutive rows having the same length. Then define p_d as the number of these rows and r_d to be their common row length. Remove these rows and repeat the process recursively on the remaining upper diagram. As Young diagrams are weakly decreasing in row length, and as each step groups maximally, no ambiguity arises in such a construction. $\square$

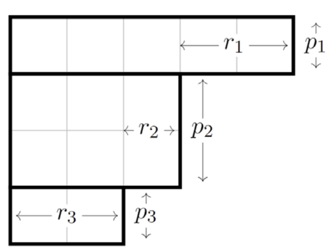

Figure 2. Multirectangular decomposition of $\lambda = (5,3,3,2)$ after reversing the rectangle order.

3.3 Polynomiality in Multirectangular Coordinates

An important result concerns the evaluations of shifted symmetric functions. They become polynomials in the multirectangular coordinates.

Theorem 3.1 (Stanley, 2001; Okounkov & Olshanski, 1997a). Let F be an α-shifted function as in Definition 2.1. When evaluated at a Young diagram λ, expressed via multirectangular coordinates (p_i, r_i), the value F_λ lies in $\mathbb{Q}(\alpha)[p_1, \ldots, p_d, r_1, \ldots, r_d]$.

Proof. By definition, each F_N is a polynomial in $\{x_1, \ldots, x_N\}$, symmetric in the shifted variables

$$x_1 - \frac{1}{\alpha}, \quad x_2 - \frac{2}{\alpha}, \quad \ldots, \quad x_N - \frac{N}{\alpha}.$$

When we evaluate F at a partition λ, we substitute $(x_1, x_2, \cdots) = (\lambda_1, \lambda_2, \cdots)$, padding with zeros if necessary. In multirectangular coordinates, the part $\lambda_i$corresponding to a row in block k can be expressed as

$$\lambda_i = r_k + r_{k+1} + \cdots + r_d.$$

Moreover, the index i (the row number) can be written explicitly in terms of the cumulative sums of the p_j. Hence, each shifted variable $\lambda_i - \frac{i}{\alpha}$ is an affine function of the p_j and r_j, with rational coefficients depending on α. Since polynomials are closed under affine substitution, and the stability property ensures compatibility across $\mathbb{N}$, it follows that $F(\lambda)$ is a polynomial in (p_i, r_i) with coefficients in $\mathbb{Q}(\alpha)$. $\square$

Finally, let us emphasize that multirectangular coordinates provide a combinatorial basis for describing shifted functions. This comes from each block p_i corresponding to a group of rows of constant cumulative width, and each width parameter r_i controls the "drop" between blocks. Hence expressing shifted Schur functions $S^\star_\mu$ and related α-shifted symmetric functions in terms of the (p_i, r_i) variables will allow us to identity polynomial expansions and a key result relating to the positivity of the shifted Schur function in the falling factorial basis.

3.4 Shifted Schur Functions

In this subsection, we focus on the case $\alpha = 1$; that is, the shifted Schur functions $S^\star_\mu$ discussed here are 1-shifted symmetric functions.

Theorem 3.2 (Alexandersson & Féray, 2017). For every $\mu \in \mathbf{Par}$, the polynomial $S^\star_\mu(\mathbf{r}^\mathbf{p})$ has nonnegative rational coefficients in the falling factorial basis $\mathfrak{B}$.

Proof. Let us fix $\mu \in \mathbf{Par}(\mathbf{n})$. Further, let *S*, *T* be set partitions of [k] with *s* and *t* parts respectively. We shall first express $S^\star_\mu$ as a sum of smaller pieces. We will then prove that each piece is nonnegative in $\mathfrak{B}$. Define the polynomials

$$A^\mu_{S,T}(x_1, \ldots, x_s, y_1, \ldots, y_t) := \sum_{\sigma,\tau \in \mathfrak{S}\square, C(\sigma) \leq S, C(\tau) \leq T} \chi^\mu(\sigma\tau)\varepsilon(\tau)\square \prod_{S_i \in S, T_j \in T} x^{i|C(\sigma)/S_i|} y^{j|C(\tau)/T_j|}$$

And define, for $\sigma, \tau \in \widetilde{\mathfrak{S}}_k$, the polynomial $N_{\sigma,\tau}(\mathbf{r}, \mathbf{p})$

$$N_{\sigma,\tau}(\mathbf{r}, \mathbf{p}) := \sum_{\substack{S,T \text{set partitions of} [k] \\ C(\sigma) \leq S, C(\tau) \leq T}} \sum_{\substack{v: S \hookrightarrow [d], w: T \hookrightarrow [d] \\ v,w \text{ compatible and injective}}} \prod_{\substack{S_i \in S \\ T_j \in T}} p_{v(S_i)}^{|C(\sigma)/S_i|} q_{w(T_j)}^{|C(\tau)/T_j|}.$$

Here $C(\sigma)$ (resp. $C(\tau)$) is the cycle partition of the permutation σ (resp. τ); χ^μ is the irreducible character of $\mathfrak{S}_k$ indexed by μ, and $\varepsilon(\tau)$ is the sign of τ.

We take for granted the fact that we can now express the polynomial

$$S^\star_\mu\left(\mathbf{r}^\mathbf{p}\right) = \frac{1}{k!} \sum\_\{\sigma,\tau \in \mathfrak{S}\square\} \chi^\mu\left(\sigma\tau\right) \varepsilon\left(\tau\right) N_{\sigma,\tau}\ (p,r)$$

Using this fact and our definition for $N_{\sigma,\tau}(\mathbf{r}, \mathbf{p})$, we may write

$$S^{\star}_{\mu}(\mathbf{r}^{\mathbf{p}}) = \frac{1}{k!} \sum_{S,T \text{set partitions of} [k]} \sum_{\substack{v:S \hookrightarrow [d], w:T \hookrightarrow [d] \\ v,w \text{ compatible} \\ \text{and injective}}} A^{\mu}_{S,T}\left(p_{v(S_1)}, \dots, p_{v(S_s)}, r_{w(T_1)}, \dots, r_{w(T_t)} \right)$$

We now notice that if $A^{\mu}_{S,T}(x_1, \dots, x_s, y_1, \dots, y_t)$ is non-negative in $\mathfrak{B}$ then so is $S^{\star}_{\mu}$, and thus it suffices to show that $A^{\mu}_{S,T}(x_1, \dots, x_s, y_1, \dots, y_t)$ is nonnegative in $\mathfrak{B}$. We appeal to the following Lemmas (see Alexandersson & Féray, 2017, pp. 15–16):

Lemma 3.2 (Alexandersson & Féray, 2017, Lemma 3.5). Fix two set partitions $S = \{S_1, \dots, S_s\}$ and $T = \{T_1, \dots, T_t\}$ of $[k]$, and let $\mu \vdash k$. Then

$$A^{\mu}_{S,T}(x_1, \dots, x_s, y_1, \dots, y_t) = \sum_{\substack{\widetilde{S} \leq S \\ \widetilde{T} \leq T}} \Big(\sum_{\substack{\sigma,\tau \in \mathfrak{S}_k \\ C(\sigma) \leq \widetilde{S} \\ C(\tau) \leq \widetilde{T}}} \chi^{\mu}(\sigma\tau) \cdot \varepsilon(\tau) \Big) \Big(\prod_{S_i \in S} (x_i)^{\left|\widetilde{S}/S_i\right|} \prod_{T_j \in T} (y_j)^{\left|\widetilde{T}/T_j\right|} \Big).$$

Lemma 3.3 (Alexandersson & Féray, 2017, Lemma 3.6) Let $\mu \vdash k$ and let S, T be set partitions of the set $[k] = \{1, \dots, k\}$. Denote by $\mathfrak{S}_S$ (resp. $\mathfrak{S}_T$) the Young subgroup of permutations whose cycles are refinements of the blocks of S (resp. T). Define

$$B^{\mu}_{S,T} = \sum_{\substack{\sigma \in \mathfrak{S}_S \\ \tau \in \mathfrak{S}_T}} \chi^{\mu}(\sigma\tau)\ \varepsilon(\tau)$$

where ε is the sign character. Then $B^{\mu}_{S,T} \geq 0$.

By the above lemmas, the polynomials $A^{\mu}_{S,T}$ have nonnegative coefficients in $\mathfrak{B}$. Since we previously expressed $S^{\star}_{\mu}(\mathbf{r}^{\mathbf{p}})$ as a non-negative linear combination of the $A^{\mu}_{S,T}$, the result follows. □

Shifted Jack Polynomials

4.1 Jack Polynomials

We recall the family of symmetric functions titled the Jack polynomials, denoted by $J^{(\alpha)}_{\mu}$ with $\mu \in \mathrm{Par}$ and parameter α. For $\alpha = 1$, Jack polynomials coincide with Schur polynomials up to multiplication by a scalar (Alexandersson & Féray, 2017).

Definition 4.1. We denote shifted Jack polynomials by $J^{\star,(\alpha)}_{\mu}$ (Okounkov & Olshanski, 1997b), where $J^{\star,(1)}_{\mu}$ is a multiple of $S^{\star}_{\mu}$ and the top degree component of $J^{\star,(\alpha)}_{\mu}$ is J_{μ}. We define the shifted Jack polynomials as the unique α-shifted symmetric function of degree at most $|\mu|$ such that for any partition λ with $|\lambda| \leq |\mu|$,

$$J^{\star,(\alpha)}_{\mu} = \begin{cases} \frac{H^{(\alpha)} H'^{(\alpha)}}{\alpha^{|\mu|}} & \text{if } \lambda = \mu, \\ 0 & \text{else.} \end{cases}$$

The proof of existence and uniqueness, as well as the top component of $J^{\star,(\alpha)}_{\mu} = J^{(\alpha)}_{\mu}$ is given in Knop and Sahi (1996).

For a diagram λ and $\mu \in \mathrm{Par}$ we define the shifted symmetric function

$$\mathfrak{K}_{\mu}(\lambda) := \begin{cases} n\Big(n-1\Big) \cdots \Big(n-k+1\Big) \frac{K^{\lambda}_{\mu,1^{n-k}}}{K^{\lambda}_{1^n}} & n \geq k \\ 0 & \text{else} \end{cases}$$

where K^{β}_{α} are the Kostka numbers. This family has an analogue for parameter α. If we consider $J^{(\alpha)}_{\lambda} = \sum_{\tau \vdash |\lambda|} \widehat{K}^{\lambda,(\alpha)}_{\tau} M_{\tau}$, then we can analogously define for a parameter α the function

$$\mathfrak{K}^{(\alpha)}_{\mu}(\lambda) := \begin{cases} \frac{1}{(n-k)!} \widehat{K}^{\lambda,(\alpha)}_{\mu,1^{n-k}} & n \geq k \\ 0 & \text{else} \end{cases}$$

where $\widehat{K}^{\lambda,(\alpha)}_{\mu 1^{n-k}}$ is the coefficient of the monomial $M_{\mu,1^{n-k}}$ in the expansion of $J^{(\alpha)}_{\lambda}$.

Definition 4.2 (Falling Factorial). We define the falling factorial to be the polynomial

$$[x]_n = (x)_n := \underbrace{x(x-1)\cdots(x-n+1)}_{\text{n factors}} = \prod_{k=1}^{n} (x-k+1) = \prod_{k=0}^{n-1} (x-k)$$

Note we interchangeably use the notation $[x]_n = (x)_n$. Moreover, we note that the falling factorials form a basis $\mathfrak{B}$ for the vector space of polynomials. We provide a brief proof below

Lemma 4.1. Let V_n be the complex vector space of polynomials of degree $\leq n$. Then $([x]_0, [x]_1, \dots, [x]_n)$ is a basis of V_n.

This follows by a triangularity argument.

Definition 4.3 (α-Falling Factorial Basis). In the polynomial ring $\mathbb{Q}[\alpha, p_1, \dots, p_d, r_1, \dots, r_d]$ we define

$$\{\ \alpha^c\ (p_1)_{a_1} \cdots (p_d)_{a_d}\ (r_1)_{b_1} \cdots (r_d)_{b_d}\ \}_{c, a_1, \dots, a_d, b_1, \dots, b_d \geq 0}$$

to be the α–*falling factorial basis* of this ring. We take for granted that this indeed forms a basis, although the proof is elementary.

Theorem 4.1 (Alexandersson & Féray, 2017, Theorem 1.5). For any $k \in \mathbb{Z}$ the quantity $J^{\star,(\alpha)}_{(k)}(\mathbf{r}^{\mathbf{p}}) = k! \mathfrak{K}^{(\alpha)}_{(k)}(\mathbf{r}^{\mathbf{p}})$ is a polynomial with non-negative rational coefficients in the α-falling factorial basis.

To prove this result, we shall appeal to the following theorem:

Theorem 4.2 (Alexandersson & Féray, 2017). For any integer $k \geq 1$ and any Young diagram λ one has,

$$\frac{1}{k!} J^{\star,(\alpha)}_{(k)}(\lambda) = \mathfrak{K}^{(\alpha)}_{(k)}(\lambda) = \sum_{\substack{A \subseteq \lambda \\ |A| = k \\ \text{column-distinct}}} \Big(\prod_{R \text{ row of } \lambda} P_{|R \cap A|}(\alpha) \Big)$$

where, for $i \geq 0$,

$$P_i(\alpha) := \prod_{j=0}^{i-1} (1 + j\alpha) \qquad (P_0(\alpha) := 1).$$

Proof of Theorem 4.1, from Alexandersson and Féray (2017). Let us first write

$$\sum_{\substack{A \subseteq \lambda \\ |A| = k \\ \text{column-distinct}}} \Big(\prod_{R \text{ row of } \lambda} P_{|R \cap A|}(\alpha) \Big) = \sum_{A \in \mathcal{CD}(\lambda, k)} w(A)$$

where we define $\mathcal{CD}(\lambda, k)$ to be the set of column-distinct subsets $A \subset \lambda$ of size k and where $w(A) := \prod_{R \text{ row of } \lambda} P_{|R \cap A|}(\alpha)$. Now to each set A we shall associate to it a skeleton $\widehat{A}$ as follows: consider $\lambda = \mathbf{r}^{\mathbf{p}}$

in rectangular coordinates, and note that every box in λ lives in a rectangle $p_i \times r_j$, where we label by i, j respectively the corresponding rows and columns. Then from λ we shall remove any rows and columns whose intersection with A is empty. This shall define our skeleton $\widehat{A}$.

Fix a skeleton $\widehat{A}_0$ and define by a_i and b_i the number of rows and columns labeled i. We observe that A column-distinct $\iff \widehat{A}$ column-distinct, and if two sets $A \neq B$ are such that $\widehat{A} = \widehat{B}$ then $w(A) = w(B)$. From this it follows that:

$$\sum_{\substack{A \in CD(r^P,k) \\ \widehat{A}=\widehat{A}A_0}} w(A) = w(A_0)|\{A \in CD\ (r^P, k)|\widehat{A} = \widehat{A}A_0\}| := \Gamma$$

It now suffices to check that for each skeleton, Γ has non-negative coefficients in $\mathfrak{B}$. Note that since $w(A_0)$ is a nonnegative polynomial in α we shall focus on the number of sets $A \subset \lambda$ with skeleton $\widehat{A}_0$.

In order to construct such a set A with shape $\lambda = \mathbf{r}^p$, we first decide the distribution of rows and columns. Let us consider $a = (a_1, \ldots, a_i, \ldots),\ b = (b_1, \ldots, b_j, \ldots)$. Then for each i we must pick rows in λ that correspond to the a_i rows in $\widehat{A} = \widehat{A}_0$. We notice that λ has p_i-many boxes labeled i and thus we have $\binom{p_i}{a_i}$ ways to choose a row. By the same logic, there are $\binom{r_i}{b_i}$ ways to choose a column, and thus we have a total of

$$\prod_i \binom{p_i}{a_i} \times \prod_i \binom{r_i}{b_i}$$

many $A \subset \lambda$ of shape $\mathbf{r}^p$ with skeleton $\widehat{A}_0$. We finally observe that $\prod_i \binom{p_i}{a_i} \times \prod_i \binom{r_i}{b_i}$ is non-negative in the basis $\mathfrak{B}$ and hence the result follows. $\square$

A Brief Comparison Between Jack- and Schur-Shifted Functions

While such results establish positivity for shifted Schur functions, corresponding to the $\alpha = 1$ case of Jack polynomials, the situation for general α-Jack polynomials remains more subtle. Determinant-style expressions are known primarily in the case of one-row partitions, $\mu = (k)$, as presented by Knop and Sahi (1996). The extensions of such expressions or positivity results to arbitrary partitions remains not fully understood.

Shifted Jack Polynomials

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